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Almost sure summability of the maximal normed partial sums of m-dependent random elements in Banach spaces

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Abstract. This paper provides sharp sufficient conditions for almost sure summability of the maximal normed partial sums of m-dependent random elements in stable type p Banach spaces, complementing recent results of Li, Qi, and Rosalsky (Trans Amer Math Soc 368(1):539–561, 2016) and Thành (Math Nachr 296(1):402–423, 2023). The main theorems are new even when the underlying random elements are independent. Our proof is direct and can be extended to other dependence structures. Two illustrative examples are also presented.

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Keywords. Maximal normed partial sum, Almost sure summability, Rademacher type p Banach space, Stable type p Banach space, (p, q)-type strong law of large numbers.

1. Introduction and the main results. Throughout, all random elements are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and take values in a real separable Banach space $\mathcal{X}$ with norm $\|\cdot\|$. The symbol C will denote a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance. For $x \in \mathbb{R}$, $\ln x$ denotes the natural logarithm of $(|x|+2)$. For a set A, $\mathbf{1}(A)$ denotes the indicator function of A.

Recently, Li et al. [5] introduced a new type of strong law of large numbers (SLLN). Let $0 < p < 2$, $q > 0$. A sequence $\{X_n, n \geq 1\}$ of random elements is said to obey the (p, q)-*type SLLN* if

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\| \sum_{i=1}^{n} X_i \|}{n^{1/p}} \right)^q < \infty \quad \text{almost surely (a.s.).} \tag{1.1}$$

If $\{X_n, n \geq 1\}$ is a sequence of independent identically distributed (i.i.d.) random elements, Li et al. [4] proved that (1.1) implies the Marcinkiewicz–Zygmund SLLN:

$$\frac{\sum_{k=1}^{n} X_k}{n^{1/p}} \to 0 \ \text{ a.s.}$$

Li et al. [5] obtained sets of necessary and sufficient conditions for the (p, q)-type SLLN for two cases: $0 < p < 1, q > p$ and $1 \leq p < 2, q \geq 1$. The results for the cases $0 < q \leq p < 1$ and $0 < q < 1 \leq p < 2$ were obtained by Thành [10]. For the case where $1 \leq p < 2$, the Banach space considered in [5,10] is required to be of stable type p. We refer to Ledoux and Talagrand [3] for definitions, equivalent characterizations, and properties of a Banach space being of Rademacher type p and of stable type p, $1 \leq p \leq 2$. The results on the (p, q)-type SLLN for the case where $1 \leq p < 2$ and $0 < q \leq p$ proved by Li et al. [5, Theorems 2.2 and 2.3] and Thành [10, Theorem 1.7] are stated in the following proposition.

Proposition 1.1. *Let $1 \leq p < 2$, $0 < q \leq p$, and let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random elements taking values in a stable type p Banach space $\mathcal{X}$. For each $n \geq 1$, let γ_n denote the quantile of order $1 - 1/n$ of $\|X_1\|$ and let $A_n = (\min\{\gamma_n^p, n\} < \|X_1\|^p \leq n)$. Then (1.1) holds if and only if*

$$\begin{cases} \mathbb{E}X_1 = 0, \ \int_0^\infty \mathbb{P}^{q/p}(\|X_1\|^q > t)\mathrm{d}t < \infty & \text{if } q < p, \\ \mathbb{E}X_1 = 0, \ \mathbb{E}(\|X_1\|^p) < \infty, \ \text{and} & \\ \sum_{n=1}^\infty n^{-1}\mathbb{E}\left(\|X_1\|^p \mathbf{1}(A_n)\right) < \infty & \text{if } 1 < q = p < 2, \\ \mathbb{E}X_1 = 0, \ \mathbb{E}(\|X_1\|^p) < \infty, & \\ \sum_{n=1}^\infty n^{-1} \left\|\mathbb{E}(X_1 \mathbf{1}(\|X_1\| \leq n))\right\| < \infty, \ \text{and} & \\ \sum_{n=1}^\infty n^{-1}\mathbb{E}\left(\|X_1\| \mathbf{1}(A_n)\right) < \infty & \text{if } q = p = 1. \end{cases} \tag{1.2}$$

It is unclear whether the methods presented in Li et al. [5] and Thành [10] can be extended to other dependence structures. The authors in [5,10] utilized various tools, including the symmetrization procedure, new versions of classical inequalities such as Lévy, Ottaviani, and Hoffmann-Jørgensen inequalities, as well as techniques developed by Hechner and Heinkel [1]. These methods heavily rely on the independence assumption. In this paper, we use a direct and simple technique to establish the (p, q)-type SLLN for sequences of m-dependent random elements in stable type p Banach spaces, complementing the aforementioned results by Li et al. [5] and Thành [10]. Our results are new even when the underlying random elements are independent and our technique is based only on a result of stochastic domination and a Rosenthal-type maximal inequality, and therefore, the obtained results can be extended to martingale differences and other dependence structures.

Let m be a nonnegative integer. A finite collection of random elements $\{X_i, 1 \leq i \leq n\}$ is said to be *m-dependent* if either $n \leq m+1$ or $n > m+1$ and the collection $\{X_1, \ldots, X_i\}$ is independent of $\{X_j, \ldots, X_n\}$ whenever $j - i > m$.

A sequence of random elements $\{X_n, n \geq 1\}$ is said to be m-dependent if for each $n \geq 1$, the collection $\{X_1, \ldots, X_n\}$ is m-dependent. The concept of m-dependence reduces to the concept of independence when $m = 0$. Also, m-dependence implies m'-dependence for every nonnegative integer $m' > m$.

Our first main theorem provides sufficient conditions for the (p, q)-type SLLN to hold for the case $1 \leq p < 2$ and $0 < q \leq p$.

Theorem 1.2. *Let $1 \leq p < 2$, $0 < q \leq p$ and let $\{X_n, n \geq 1\}$ be a sequence of m-dependent random elements taking values in a stable type p Banach space. If*

$$\mathbb{E}X_n \equiv 0 \quad \text{and} \quad \sup_{n \geq 1} \mathbb{E}\left(\|X_n\|^p \ln^\delta \|X_n\|\right) < \infty \text{ for some } \delta \geq 2/q, \tag{1.3}$$

then

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\max_{1 \leq k \leq n} \|\sum_{i=1}^{k} X_i\|}{n^{1/p}} \right)^q < \infty \quad a.s. \tag{1.4}$$

The following remark compares Theorem 1.2 and Proposition 1.1.

Remark 1.3. (i) In Theorem 1.2, we do not require the random elements to be identically distributed.

(ii) The moment conditions in (1.3) are much simpler than those in (1.2). Even when the random elements are i.i.d, the second half of (1.3) can not be weakened to $\mathbb{E}\|X_1\|^p < \infty$ (see [5, Example 4.3]).

(iii) Our conclusion (1.4) is stronger than (1.1). From (1.4), we can easily obtain the Marcinkiewicz–Zygmund SLLN without assuming any dependence structure or stochastic domination condition (see Proposition 2.3 in Sect. 2). Li et al. [4] employed an elegant and meticulous argument to prove that (1.1) also guarantees the Marcinkiewicz-Zygmund SLLN. Their result assumes that the random elements are i.i.d. In Thành [10], the author presented another proof of the Li et al. result, and was able to remove the identical distribution assumption which plays an important role in the proof of Li et al. [5]. However, Thành [10] had to assume that the random elements are stochastically dominated by a real-valued random variable X with $\mathbb{E}|X|^p < \infty$. From this perspective, (1.4) proves to be more useful than (1.1).

The next theorem considers the case where $1 \leq p < 2$ and $q > p$.

Theorem 1.4. *Let $1 \leq p < 2$, $q > p$ and let $\{X_n, n \geq 1\}$ be a sequence of m-dependent random elements taking values in a stable type p Banach space $\mathcal{X}$. If*

$$\mathbb{E}X_n \equiv 0 \quad \text{and} \quad \sup_{n \geq 1} \mathbb{E}\left(\|X_n\|^p \ln^\delta \|X_n\|\right) < \infty \text{ for some } \delta > 1, \tag{1.5}$$

then (1.4) holds.

The following example shows that Theorems 1.2 and 1.4 do not hold in general if $p = 2$.

Example 1.5. Let $p=2$ and consider the case where $\mathcal{X}$ is the real line which is of stable type 2. Let $\{X_n, n\ge 1\}$ be a sequence of independent standard normal distribution random variables. It is clear that both (1.3) and (1.5) are satisfied. For all $q>0$, we have

$$\mathbb{P}\left(\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\max_{1\le k\le n}|\sum_{i=1}^{k}X_i|}{n^{1/2}}\right)^q=\infty\right)\ge \mathbb{P}\left(\sum_{n=1}^{\infty}\frac{|\sum_{i=1}^{n}X_i|^q}{n^{1+q/2}}=\infty\right). \tag{1.6}$$

Since

$$\left(\left|\sum_{i=1}^{n}X_i\right|\ge n^{1/2}\right)\subset\left(\sum_{n=1}^{\infty}\frac{|\sum_{i=1}^{n}X_i|^q}{n^{1+q/2}}=\infty\right),$$

it follows from (1.6) that

$$\begin{aligned}\mathbb{P}\left(\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\max_{1\le k\le n}|\sum_{i=1}^{k}X_i|}{n^{1/2}}\right)^q=\infty\right)&\ge \mathbb{P}\left(\left|\sum_{i=1}^{n}X_i\right|\ge n^{1/2}\right)\\ &=\mathbb{P}(|X_1|\ge 1)>0\end{aligned}$$

showing that (1.4) fails with $p=2$.

The next example shows that Theorems 1.2 and 1.4 can fail if the stable type p hypothesis is weakened to the Rademacher type p hypothesis.

Example 1.6. Let $1\le p<2$ and let $\mathcal{X}$ be the real separable Banach space ℓ_p of absolute p-th power summable real sequences $x=(x_1,x_2,\ldots)$ with norm

$$\|x\|=\left(\sum_{j=1}^{\infty}|x_j|^p\right)^{1/p}.$$

The Banach space ℓ_p is of Rademacher type p but is not of stable type p. For $n\ge 1$, let e_n denote the element in ℓ_p having 1 for its n-th coordinate and 0 for the other coordinates. Let $\{X_n, n\ge 1\}$ be a sequence of independent random elements in ℓ_p such that

$$\mathbb{P}(X_n=e_n)=\mathbb{P}(X_n=-e_n)=1/2,\ n\ge 1.$$

Then $\|X_n\|\equiv 1$ a.s., $n\ge 1$, and therefore both (1.3) and (1.5) are satisfied. For all $q>0$, (1.4) fails since with probability 1

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\max_{1\le k\le n}\|\sum_{i=1}^{k}X_i\|}{n^{1/p}}\right)^q&\ge\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\|\sum_{i=1}^{n}X_i\|}{n^{1/p}}\right)^q\\ &=\sum_{n=1}^{\infty}\frac{1}{n}=\infty.\end{aligned}$$

2. Proof of the main results. Firstly, we will present two lemmas which are needed in the proof of the main results. The first lemma is a Rosenthal-type inequality for maximal normed partial sums of m-dependent random elements in Rademacher type p Banach spaces. The proof is done by separating the underlying random elements into $(m+1)$ groups of independent random elements (see, e.g., [9, Lemma 2.2] or [11, Lemma 3.3]) and then using a Rosenthal-type inequality for maximal normed partial sums of independent random elements (see [7, Lemma 2.1]).

Lemma 2.1. *Let $1 \le p \le 2$ and $\{X_i, 1 \le i \le n\}$ be a family of m-dependent mean zero random elements in a Rademacher type p Banach space $\mathcal{X}$. Then for all $q \ge p$, there exists a constant $C_{p,q,m} \in (0,\infty)$ depending only on p, q, and m such that*

$$\mathbb{E}\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} X_i\right\|^{q}\right) \le C_{p,q,m}\left(\left(\sum_{i=1}^{n}\mathbb{E}\|X_i\|^{p}\right)^{q/p} + \sum_{i=1}^{n}\mathbb{E}\|X_i\|^{q}\right). \quad (2.1)$$

The second lemma was proved recently by Rosalsky and Thành [6]. It shows that bounded moment type conditions on a sequence of real-valued random variables $\{X_n, n \ge 1\}$ can accomplish stochastic domination.

Lemma 2.2. *Let $\{X_i, i \in I\}$ be a family of real-valued random variables. If*

$$\sup_{i\in I}\mathbb{E}(|X_i|^p \ln^{1+\varepsilon}|X_i|) < \infty \text{ for some } p > 0 \text{ and for some } \varepsilon > 0,$$

then there exists a real-valued random variable X such that $\mathbb{E}|X|^p < \infty$ and $\{X_i, i \in I\}$ is stochastically dominated by X, that is,

$$\sup_{i\in I}\mathbb{P}(|X_i| > x) \le \mathbb{P}(|X| > x) \text{ for all } x \in \mathbb{R}.$$

We will now present the proof of Theorems 1.2 and 1.4.

Proof of Theorem 1.2. For $n \ge 1$, set

$$Y_n = X_n \mathbf{1}(\|X_n\| \le n^{1/p}),$$
$$T_n = \sum_{i=1}^{n}(Y_i - \mathbb{E}Y_i).$$

Since $0 < q \le p < 2$ and $\delta \ge 2/q$, we have $\delta > 1$. Applying (1.3) and Lemma 2.2, we conclude that the sequence of real-valued random variables $\{\|X_n\|, n \ge 1\}$ is stochastically dominated by a real-valued random variable X satisfying $\mathbb{E}|X|^p < \infty$. It thus follows that

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbb{P}(X_n \ne Y_n) &= \sum_{n=1}^{\infty}\mathbb{P}(\|X_n\| > n^{1/p})\\ &\le \sum_{n=1}^{\infty}\mathbb{P}(|X| > n^{1/p})\\ &\le 1 + \mathbb{E}|X|^p < \infty. \end{aligned} \quad (2.2)$$

By (2.2) and the Borel–Cantelli lemma, we have $\mathbb{P}(X_n \neq Y_n \text{ i.o.}) = 0$. Therefore, to obtain (1.4), it suffices to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\max_{1\le k\le n} \|\sum_{i=1}^{k} Y_i\|}{n^{1/p}} \right)^q < \infty \quad \text{a.s.} \tag{2.3}$$

Since $\sum_{i=1}^{k} Y_i = T_k + \sum_{i=1}^{k} \mathbb{E}Y_i$ for all $k \ge 1$, (2.3) will be proved if we can show that

$$\sum_{n=1}^{\infty} \frac{\max_{1\le k\le n} \|T_k\|^q}{n^{1+q/p}} < \infty \quad \text{a.s.}, \tag{2.4}$$

and

$$\sum_{n=1}^{\infty} \frac{\max_{1\le k\le n} \|\sum_{i=1}^{k} \mathbb{E}Y_i\|^q}{n^{1+q/p}} < \infty. \tag{2.5}$$

Since $\mathcal{X}$ is of stable type p, it is of Rademacher type $p+\varepsilon$ for some $0 < \varepsilon < 2-p$. We can assume, without loss of generality, that the function $f(x) = x^{\varepsilon} \ln^{-\delta} x$ is strictly increasing on $[1,\infty)$. (In the general case, we can let A be a large positive number and consider the function $x^{\varepsilon} \ln^{-\delta}(x+A)$ instead). Therefore,

$$\begin{aligned}
\mathbb{E}\left(\max_{1\le k\le n} \|T_k\|^q \right) &\le \left(\mathbb{E}\left(\max_{1\le k\le n} \|T_k\|^{p+\varepsilon} \right) \right)^{q/(p+\varepsilon)} \\
&\le C \left(\sum_{i=1}^{n} \mathbb{E}\|Y_i - \mathbb{E}Y_i\|^{p+\varepsilon} \right)^{q/(p+\varepsilon)} \\
&\le C \left(\sum_{i=1}^{n} \mathbb{E}\|Y_i\|^{p+\varepsilon} \right)^{q/(p+\varepsilon)} \\
&\le C \left(\sum_{i=1}^{n} i^{\varepsilon/p} \ln^{-\delta} i \mathbb{E}\|Y_i\|^p \ln^{\delta} \|Y_i\| \right)^{q/(p+\varepsilon)} \\
&\le C \left(\sum_{i=1}^{n} i^{\varepsilon/p} \ln^{-\delta} i \right)^{q/(p+\varepsilon)} \\
&\le C n^{q/p} \ln^{-\delta q/(p+\varepsilon)} n,
\end{aligned} \tag{2.6}$$

where we have applied Jensen's inequality in the first inequality, Lemma 2.1 in the second inequality, and (1.3) in the fifth inequality. Since $\delta q/(p+\varepsilon) > \delta q/2 \ge 1$, it follows from (2.6) that

$$\mathbb{E}\left(\sum_{n=1}^{\infty} \frac{\max_{1\le k\le n} \|T_k\|^q}{n^{1+q/p}} \right) < \infty,$$

which immediately implies (2.4).

Applying the assumption $\mathbb{E}X_i \equiv 0$ and (1.3) again, we have

$$\begin{aligned}\max_{1\le k\le n}\left\|\sum_{i=1}^{k}\mathbb{E}Y_i\right\|^q &\le \left(\sum_{i=1}^{n}\|\mathbb{E}(X_i\mathbf{1}(\|X_i\|>i^{1/p}))\|\right)^q\\ &\le \left(\sum_{i=1}^{n}\mathbb{E}(\|X_i\|\mathbf{1}(\|X_i\|>i^{1/p}))\right)^q\\ &\le C\left(\sum_{i=1}^{n}i^{1/p-1}\ln^{-\delta} i\,\mathbb{E}(\|X_i\|^p\ln^{\delta}\|X_i\|)\right)^q\\ &\le C\left(\sum_{i=1}^{n}i^{1/p-1}\ln^{-\delta} i\right)^q\\ &\le Cn^{q/p}\ln^{-\delta q} n. \qquad (2.7)\end{aligned}$$

Since $\delta q > \delta q/2 \ge 1$, (2.5) follows from (2.7). The proof is completed. □

Proof of Theorem 1.4. Let Y_n and T_n be as in the proof of Theorem 1.2. Using the same argument as in the proof of Theorem 1.2, it remains to prove that (2.4) and (2.5) hold. Since $\mathcal{X}$ is of stable type $p<2$ and $q>p$, it is of Rademacher type $p+\varepsilon$ for some $0<\varepsilon<\min\{q-p,2-p\}$. By applying Lemma 2.1, we obtain

$$\begin{aligned}\mathbb{E}\left(\max_{1\le k\le n}\|T_k\|^q\right) &\le C\left(\left(\sum_{i=1}^{n}\mathbb{E}\|Y_i-\mathbb{E}Y_i\|^{p+\varepsilon}\right)^{q/(p+\varepsilon)}+\sum_{i=1}^{n}\mathbb{E}\|Y_i-\mathbb{E}Y_i\|^q\right)\\ &\le C\left(\left(\sum_{i=1}^{n}\mathbb{E}\|Y_i\|^{p+\varepsilon}\right)^{q/(p+\varepsilon)}+\sum_{i=1}^{n}\mathbb{E}\|Y_i\|^q\right)\\ &:= I_1+I_2. \qquad (2.8)\end{aligned}$$

Similar to the proof of Theorem 1.2, it suffices to consider the case where the function $f(x)=x^{\varepsilon}\ln^{-\delta}x$ is strictly increasing on $[1,\infty)$. By (1.5), we have

$$\begin{aligned}I_1 &\le C\left(\sum_{i=1}^{n}i^{\varepsilon/p}\ln^{-\delta} i\,\mathbb{E}\|Y_i\|^p\ln^{\delta}\|Y_i\|\right)^{q/(p+\varepsilon)}\\ &\le C\left(\sum_{i=1}^{n}i^{\varepsilon/p}\ln^{-\delta} i\right)^{q/(p+\varepsilon)}\\ &\le Cn^{q/p}\ln^{-\delta q/(p+\varepsilon)} n \le Cn^{q/p}\ln^{-\delta} n, \qquad (2.9)\end{aligned}$$

and

$$\begin{aligned}I_2 &\le C\sum_{i=1}^{n}i^{(q-p)/p}\ln^{-\delta} i\,\mathbb{E}\|X_i\|^p\ln^{\delta}\|X_i\|\\ &\le C\sum_{i=1}^{n}i^{(q-p)/p}\ln^{-\delta} i\\ &\le Cn^{q/p}\ln^{-\delta} n. \qquad (2.10)\end{aligned}$$

Combining (2.8)–(2.10) yields

$$\mathbb{E}\left(\sum_{n=1}^{\infty}\frac{\max_{1\le k\le n}\|T_k\|^q}{n^{1+q/p}}\right)<\infty,$$

which immediately implies (2.4). By (2.7), (2.5) holds. The proof of the theorem is completed. □

From (1.4), one can easily obtain the Marcinkiewicz–Zygmund SLLN without assuming any dependence structure or stochastic domination condition. More generally, we have the following result.

Proposition 2.3. *Let $\{X_n, n\ge 1\}$ be a sequence of random elements and let $\{u_n, n\ge 1\}$ be an increasing sequence of positive real numbers satisfying*

$$\sup_{j\ge 0}\frac{u_{2^{j+1}}}{u_{2^j}}<\infty. \tag{2.11}$$

If

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\max_{1\le k\le n}\|\sum_{i=1}^{k}X_i\|}{u_n}\right)^q<\infty \quad \textit{a.s. for some } q>0, \tag{2.12}$$

then

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}X_i}{u_n}=0 \ \textit{a.s.} \tag{2.13}$$

Proof. Using (2.11) and (2.12), we have

$$\begin{aligned}\sum_{j=0}^{\infty}\left(\frac{\max_{1\le k\le 2^j}\|\sum_{i=1}^{k}X_i\|}{u_{2^j}}\right)^q &\le C\sum_{j=0}^{\infty}\sum_{n=2^j}^{2^{j+1}-1}\frac{1}{n}\left(\frac{\max_{1\le k\le 2^j}\|\sum_{i=1}^{k}X_i\|}{u_{2^{j+1}}}\right)^q\\ &\le C\sum_{j=0}^{\infty}\sum_{n=2^j}^{2^{j+1}-1}\frac{1}{n}\left(\frac{\max_{1\le k\le n}\|\sum_{i=1}^{k}X_i\|}{u_n}\right)^q\\ &= C\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\max_{1\le k\le n}\|\sum_{i=1}^{k}X_i\|}{u_n}\right)^q<\infty \ \text{ a.s.,}\end{aligned}$$

which immediately implies

$$\lim_{j\to\infty}\frac{\max_{1\le k\le 2^j}\left\|\sum_{i=1}^{k}X_i\right\|}{u_{2^j}}=0 \text{ a.s.} \tag{2.14}$$

We infer from (2.11) and (2.14) that (2.13) holds. □

3. The (p,q)-type SLLN for negatively associated random variables. The concept of negative association of (real-valued) random variables was introduced by Joag-Dev and Proschan [2]. A collection $\{X_1,\dots,X_n\}$ of random variables is said to be negatively associated if for any disjoint subsets A, B of $\{1,\dots,n\}$ and any real coordinatewise nondecreasing functions f on $\mathbb{R}^{|A|}$ and g on $\mathbb{R}^{|B|}$,

$$\operatorname{Cov}(f(X_k, k\in A), g(X_k, k\in B))\le 0$$

whenever the covariance exists, where $|A|$ denotes the cardinality of A. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be negatively associated if for each $n \geq 1$, the collection $\{X_1, \ldots, X_n\}$ is negatively associated.

Negatively associated random variables have gained significant popularity owing to their extensive applications. Since the Rosenthal maximal inequality still holds under negative association (see Shao [8, Theorem 2]), we have the following result on the (p, q)-type SLLN for sequences of negatively associated random variables. To our best knowledge, this is the first result on the (p, q)-type SLLN for dependent random variables.

Theorem 3.1. *Let $1 \leq p < 2$, $q > 0$ and let $\{X_n, n \geq 1\}$ be a sequence of negatively associated mean zero random variables. If*

$$\begin{cases} \sup_{n\geq 1} \mathbb{E}\left(|X_n|^p \ln^\delta |X_n|\right) < \infty \text{ for some } \delta \geq 2/q & \text{if } q \leq p, \\ \sup_{n\geq 1} \mathbb{E}\left(|X_n|^p \ln^\delta |X_n|\right) < \infty \text{ for some } \delta > 1 & \text{if } q > p, \end{cases}$$

then $\{X_n, n \geq 1\}$ obeys the (p, q)-type SLLN

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i|}{n^{1/p}} \right)^q < \infty \quad a.s.$$

Proof. For $n \geq 1$, set

$$Y_n = -n^{1/p}\mathbf{1}(X_n < -n^{1/p}) + X_n\mathbf{1}(|X_n| \leq n^{1/p}) + n^{1/p}\mathbf{1}(X_n > n^{1/p}).$$

Then the sequence $\{Y_n - \mathbb{E}Y_n, n \geq 1\}$ is negatively associated (see Joag-Dev and Proschan [2, Property 6]). The rest of the proof is the same as that of Theorems 1.2 and 1.4, with only some minor adjustments. □

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